TWMS Jour. Pure Appl. Math., V.3, N.2, 2012, pp.190-201

SOME GENERALIZED SEQUENCE SPACES DEFINED BY A MUSIELAK-ORLICZ FUNCTION OVER n-NORMED SPACES

KULDIP RAJ[1], SUNIL K. SHARMA[1]

Abstract. In the present paper we introduce some sequence spaces defined by a Musielak-Orlicz function $\mathcal{M} = (M_k)$ over n-normed spaces. We study some topological properties and prove some inclusion relations between these spaces.

Keywords: Paranorm space, difference sequence space, Orlicz function, Musielak-Orlicz function, n-normed space.

AMS Subject Classification: 40A05, 46A45.

1. Introduction

The concept of 2-normed spaces was initially developed by Gähler [2] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [12]. Since then, many others have studied this concept and obtained various results, see Gunawan ([3], [4]) and Gunawan and Mashadi [5] and references therein.

Let w be the set of all sequences of real or complex numbers and l_∞, c and c_0 be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$, respectively.

A sequence $x \in l_\infty$ is said to be almost convergent if all Banach limits of x coincide. Lorentz [8] proved that

$$\hat{c} = \Big\{x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} x_{k+s} \quad \text{exists, uniformly in } \; s\Big\}.$$

Maddox ([9], [10]) has defined x to be strongly almost convergent to a number L if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L| = 0, \;\; \text{uniformly in } \; s.$$

Let $p = (p_k)$ be a sequence of strictly positive real numbers. Nanda [14] has defined the following sequence spaces :

$$[\hat{c}, p] = \Big\{x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L|^{p_k} = 0, \;\; \text{uniformly in } \; s\Big\},$$

$$[\hat{c}, p]_0 = \Big\{x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+s}|^{p_k} = 0, \;\; \text{uniformly in } \; s\Big\}$$

and

$$[\hat{c}, p]_\infty = \Big\{x = (x_k) : \sup_{s,n} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s}|^{p_k} < \infty\Big\}.$$

The notion of difference sequence spaces was introduced by Kizmaz [6], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et

[1] School of Mathematics, Shri Mata Vaishno Devi University, Katra, India
e-mail: kuldeepraj68@rediffmail.com, sunilksharma42@yahoo.co.in
Manuscript received September 2011.

and Colak [1] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let m, r be non-negative integers, then for $Z = l_\infty$, c and c_0, we have sequence spaces,

$$Z(\Delta_r^m) = \{x = (x_k) \in w : (\Delta_r^m x_k) \in Z\},$$

where $\Delta_r^m x = (\Delta_r^m x_k) = (\Delta_r^{m-1} x_k - \Delta_r^{m-1} x_{k+r})$ and $\Delta_r^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_r^m x_k = \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+rv}.$$

Taking $m = r = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [6].

An Orlicz function M is a function, which is continuous and convex with $M(0) = 0$, $M(q) > 0$ for $q > 0$ and $M(q) \longrightarrow \infty$ as $q \longrightarrow \infty$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to define the sequence space, then the space

$$\ell_M = \Big\{x = (x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\},$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [10] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function see ([11],[13]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - (M_k)(u) : u \geq 0\},\ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

For more details about sequence spaces see ([15], [16], [17], [18]) and many others.

2. Preliminaries

Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

(1) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
(2) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
(3) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
(4) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E =$ the volume of the n-dimensional parallelepiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$ and $||.||_E$ is the n-norm on Euclidean space $\mathbb{R}^n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$, for all $x \in X$;
(2) $p(-x) = p(x)$, for all $x \in X$;
(3) $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$;
(4) if (σ_n) is a sequence of scalars with $\sigma_n \to \sigma$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\sigma_n x_n - \sigma x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [19], Theorem 10.4.2, p.183).

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space, $p = (p_k)$ be bounded sequence of strictly positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. By $S(n - X)$ we denote the space of all sequences defined over $(X, ||\cdot, \cdots, \cdot||)$. In the present paper we define the following sequence spaces:

$$\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big](\Delta_r^m) =$$

$$= \Big\{x = (x_k) \in S(n - X) : \lim_\eta \frac{1}{\eta} \sum_{k=1}^{\eta} \Big[M_k\Big(||\frac{u_k \Delta_r^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0,$$

$$\text{uniformly in } s, \text{ for some } L \text{ and } \rho > 0 \quad \Big\},$$

$$\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m) =$$

$$= \Big\{x = (x_k) \in S(n-X) : \lim_{\eta} \frac{1}{\eta} \sum_{k=1}^{\eta} \Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0, \text{ uniformly in } s, \text{ for } \rho > 0 \Big\},$$

and

$$\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_{\infty}(\Delta_r^m) =$$

$$= \Big\{x = (x_k) \in S(n-X) : \sup_{s,\eta} \frac{1}{\eta} \sum_{k=1}^{\eta} \Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < \infty, \text{ for } \rho > 0\Big\}.$$

If we take $\mathcal{M}(x) = x$, we get

$$\Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big](\Delta_r^m) =$$

$$= \Big\{x = (x_k) \in S(n-X) : \lim_{\eta} \frac{1}{\eta} \sum_{k=1}^{\eta} \Big(||\frac{u_k\Delta_r^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} = 0, \text{ uniformly in } s, \text{ for some } L \text{ and } \rho > 0 \Big\},$$

$$\Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m) =$$

$$= \Big\{x = (x_k) \in S(n-X) : \lim_{\eta} \frac{1}{\eta} \sum_{k=1}^{\eta} \Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} = 0, \text{ uniformly in } s, \text{ for } \rho > 0 \Big\},$$

and

$$\Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_{\infty}(\Delta_r^m) =$$

$$= \Big\{x = (x_k) \in S(n-X) : \sup_{s,\eta} \frac{1}{\eta} \sum_{k=1}^{\eta} \Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} < \infty, \text{ for } \rho > 0\Big\}.$$

If we take $p = (p_k) = 1$ for all $k \in \mathbb{N}$, we get

$$\Big[\hat{c}, \mathcal{M}, u, ||\cdot, \cdots, \cdot||\Big](\Delta_r^m) =$$

$$= \Big\{x = (x_k) \in S(n-X) : \lim_{\eta} \frac{1}{\eta} \sum_{k=1}^{\eta} \Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0, \text{ uniformly in } s, \text{ for some } L \text{ and } \rho > 0 \Big\},$$

$$\Big[\hat{c}, \mathcal{M}, u, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m) =$$

$$= \Big\{x = (x_k) \in S(n-X) : \lim_{\eta} \frac{1}{\eta} \sum_{k=1}^{\eta} \Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0, \text{ uniformly in } s, \text{ for } \rho > 0 \Big\},$$

and

$$\Big[\hat{c}, \mathcal{M}, u, ||\cdot, \cdots, \cdot||\Big]_{\infty}(\Delta_r^m) =$$

$$= \Big\{x = (x_k) \in S(n-X) : \sup_{s,\eta} \frac{1}{\eta} \sum_{k=1}^{\eta} \Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] < \infty, \text{ for } \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \qquad (1)$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main aim of this paper is to study some sequence spaces defined by a Musielak-Orlicz function over n-normed spaces. We also make an effort to study some topological properties and some inclusion relations between these spaces.

3. Main results

Theorem 3.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. Then the spaces $\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big](\Delta_r^m)$, $\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$ and $\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m)$ are linear spaces.*

Proof. Let $x = (x_k)$, $y = (y_k) \in \Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$ and α, β be any scalars. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_\eta \frac{1}{\eta} \sum_{k=1}^{\eta} \Big[M_k\Big(||\frac{u_k \Delta_r^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0$$

and

$$\lim_\eta \frac{1}{\eta} \sum_{k=1}^{\eta} \Big[M_k\Big(||\frac{u_k \Delta_r^m y_{k+s}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is non-decreasing convex function and so by using inequality (1), we have

$$\frac{1}{\eta} \sum_{k=1}^{\eta} \Big[M_k\Big(||\frac{u_k \Delta_r^m (\alpha x_{k+s} + \beta y_{k+s})}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq$$

$$\leq \frac{1}{\eta} \sum_{k=1}^{\eta} \Big[M_k\Big(||\frac{u_k \Delta_r^m \alpha x_{k+s}}{\rho_3}, z_1, \cdots, z_{n-1}|| + ||\frac{u_k \Delta_r^m \beta y_{k+s}}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq$$

$$\leq \frac{1}{\eta} \sum_{k=1}^{\eta} \Big[M_k\Big(||\frac{u_k \Delta_r^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}|| + ||\frac{u_k \Delta_r^m y_{k+s}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq$$

$$\leq K \frac{1}{\eta} \sum_{k=1}^{\eta} \Big[M_k\Big(||\frac{u_k \Delta_r^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} +$$

$$+ K \frac{1}{\eta} \sum_{k=1}^{\eta} \Big[M_k\Big(||\frac{u_k \Delta_r^m y_{k+s}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \to 0$$

$$\text{as } n \to \infty, \text{ uniformly in } s.$$

So that $\alpha x + \beta y \in \Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$. Thus $\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$ is a linear space. Similarly, we can prove that $\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m)$ and $\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big](\Delta_r^m)$ are linear spaces. □

Theorem 3.2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. Then*

$\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$ *is a paranormed space with respect to the paranorm defined by*

$$g(x) = \inf\Big\{\rho^{\frac{p_n}{H}} : \Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\},$$

where $H = \max(1, \sup_k p_k < \infty)$.

Proof. Clearly $g(x) \geq 0$ for $x = (x_k) \in \Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$. Since $M_k(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then

$$\inf\Big\{\rho^{\frac{p_n}{H}} : \Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\} = 0.$$

This implies that for a given $\epsilon > 0$, there exists some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho_\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1.$$

Thus

$$\Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq$$

$$\leq \quad \Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho_\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1,$$

for each n. Suppose that $x_k \neq 0$ for each $k \in N$. This implies that $u_k\Delta_r^m x_{k+s} \neq 0$, for each $k, s \in N$. Let $\epsilon \to 0$, then $||\frac{u_k\Delta_r^m x_{k+s}}{\epsilon}, z_1, \cdots, z_{n-1}|| \to \infty$. It follows that

$$\Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \to \infty$$

which is a contradiction. Therefore, $u_k\Delta^m x_{k+s} = 0$ for each k and thus $x_k = 0$ for each $k \in N$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1$$

and

$$\Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m y_{k+s}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1$$

for each n. Let $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we have

$$\Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m (x_{k+s} + y_{k+s})}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq$$

$$
\begin{aligned}
&\leq \Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}+\Delta_r^m y_{k+s}}{\rho_1+\rho_2}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq \\
&\leq \Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[\frac{\rho_1}{\rho_1+\rho_2}M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho_1}, z_1,\cdots,z_{n-1}||\Big) + \\
&+ \frac{\rho_2}{\rho_1+\rho_2}M_k\Big(||\frac{u_k\Delta_r^m y_{k+s}}{\rho_2}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq \\
&\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho_1}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} + \\
&+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m y_{k+s}}{\rho_2}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1.
\end{aligned}
$$

Since ρ's are non-negative, so we have

$$
\begin{aligned}
g(x+y) &= \inf\Big\{\rho^{\frac{p_n}{H}} : \Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}+u_k\Delta_r^m y_{k+s}}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\} \leq \\
&\leq \inf\Big\{\rho_1^{\frac{p_n}{H}} : \Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\ \Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho_1}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\} + \\
&+ \inf\Big\{\rho_2^{\frac{p_n}{H}} : \Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m y_{k+s}}{\rho_2}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\}.
\end{aligned}
$$

Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = \inf\Big\{\rho^{\frac{p_n}{H}} : \Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m \lambda x_{k+s}}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

Then

$$g(\lambda x) = \inf\Big\{(|\lambda|t)^{\frac{p_n}{H}} : \Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{t}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

where $t = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_n} \leq \max(1, |\lambda|^{\sup p_r})$, we have

$$g(\lambda x) \leq \max(1, |\lambda|^{\sup p_n}) \inf\Big\{t^{\frac{p_n}{H}} : \Big(\frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{t}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. □

Theorem 3.3. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent*

(i) $\Big[\hat{c}, u, p, ||\cdot,\cdots,\cdot||\Big]_\infty(\Delta_r^m) \subseteq \Big[\hat{c}, \mathcal{M}, u, p, ||\cdot,\cdots,\cdot||\Big]_\infty(\Delta_r^m)$,

(ii) $\Big[\hat{c}, u, p, ||\cdot,\cdots,\cdot||\Big]_0(\Delta_r^m) \subseteq \Big[\hat{c}, \mathcal{M}, u, p, ||\cdot,\cdots,\cdot||\Big]_\infty(\Delta_r^m)$,

(iii) $\sup_n \frac{1}{\eta}\sum_{k=1}^{\eta}[M_k(t)]^{p_k} < \infty$, *where* $t = ||\frac{u_k\Delta_r^m x_{k+s}}{\rho}, z_1,\cdots,z_{n-1}|| > 0$.

Proof. (i) $\implies$ (ii) is obvious, since $\Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m) \subseteq \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m)$.

(ii) $\implies$ (iii). Suppose $\Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m) \subseteq \Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m)$ and let (iii) does not hold. Then for some $t > 0$

$$\sup_\eta \frac{1}{\eta}\sum_{k=1}^{\eta}[M_k(t)]^{p_k} = \infty,$$

and therefore there is a sequence (η_i) of positive integers such that

$$\frac{1}{\eta_i}\sum_{k=1}^{\eta_i}[M_k(i^{-1})]^{p_k} > i, \quad i = 1, 2, ... \tag{2}$$

Define $x = (x_k)$ by

$$x_k = \begin{cases} i^{-1}, & 1 \le k \le \eta_i, \quad i = 1, 2, ... \\ 0, & k \ge \eta_i. \end{cases}$$

Then $x = (x_k) \in \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$ but $x = (x_k) \notin \Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m)$ which contradicts (ii). Hence (iii) must hold.

(iii) $\implies$ (i). Suppose $x = (x_k) \in \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m)$ and $x = (x_k) \notin \Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m)$.

Then

$$\sup_{s,\eta} \frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k\Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = \infty. \tag{3}$$

Let $t = ||\frac{u_k\Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||$ for each k and fixed s, then by (3)

$$\sup_\eta \frac{1}{\eta}\sum_{k=1}^{\eta}[M_k(t)]^{p_k} = \infty,$$

which contradicts (iii). Hence (i) must hold. □

Theorem 3.4. *Let $1 \le p_k \le \sup_k p_k < \infty$. Then the following statements are equivalent*

(i) $\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m) \subseteq \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$,

(ii) $\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m) \subseteq \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m)$,

(iii) $\inf_\eta \frac{1}{\eta}\sum_{k=1}^{\eta}[M_k(t)]^{p_k} > 0, \quad t > 0.$

Proof. (i) $\implies$ (ii) is obvious.

(ii) $\implies$ (iii) Suppose $\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m) \subseteq \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m)$ and let (iii) does not hold. Then

$$\inf_\eta \frac{1}{\eta}\sum_{k=1}^{\eta}[M_k(t)]^{p_k} = 0, \quad t > 0. \tag{4}$$

We can choose an index sequence (η_i) such that

$$\frac{1}{\eta_i}\sum_{k=1}^{\eta_i}[M_k(i)]^{p_k} < i^{-1}, \quad i = 1, 2, ...$$

Define the sequence $x = (x_k)$ by

$$(x_k) = \begin{cases} i, & 1 \le k \le \eta_i, \quad i = 1, 2, ... \\ 0, & k \ge \eta_i. \end{cases}$$

Thus by (4), $x = (x_k) \in \Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$ but $x = (x_k) \notin \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m)$ which contradicts (ii). Hence (iii) must hold.

(iii) $\Longrightarrow$ (i) Let $x = (x_k) \in \Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$. That is,

$$\lim_\eta \frac{1}{\eta} \sum_{k=1}^{\eta} \Big[M_k\Big(||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0, \text{ uniformly in } s. \tag{5}$$

Suppose (iii) hold and $x = (x_k) \notin \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$. Then for some number $\epsilon_0 > 0$ and index η_0, we have $||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}|| \ge \epsilon_0$, for some $s > s'$ and $1 \le k \le \eta_0$. Therefore

$$[M_k(\epsilon_0)]^{p_k} \le \Big[M_k\Big(||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$

and consequently by (5)

$$\lim_\eta \frac{1}{\eta} \sum_{k=1}^{\eta} [M_k(\epsilon_0)]^{p_k} = 0,$$

which contradicts (iii). Hence $\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m) \subseteq \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$. □

Theorem 3.5. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Let $1 \le p_k \le \sup_k p_k < \infty$. Then*

$$\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m) \subseteq \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$$

holds if and only if

$$\lim_\eta \frac{1}{\eta} \sum_{k=1}^{\eta} [M_k(t)]^{p_k} = \infty, \quad t > 0. \tag{6}$$

Proof. Suppose $\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m) \subseteq \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$ and let (6) does not hold. Therefore there is a number $t_0 > 0$ and an index sequence (η_i) such that

$$\frac{1}{\eta_i} \sum_{k=1}^{\eta_i} [M_k(t_0)]^{p_k} \le N < \infty, \ i = 1, 2, ... \tag{7}$$

Define the sequence $x = (x_k)$ by

$$(x_k) = \begin{cases} t_0, & 1 \le k \le \eta_i, \quad i = 1, 2, ... \\ 0, & k \ge \eta_i. \end{cases}$$

Clearly, $x = (x_k) \in \Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m)$ but $x = (x_k) \notin \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$. Hence (6) must hold.

Conversely, if $x = (x_k) \in \Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m)$, then for each s and η

$$\frac{1}{\eta} \sum_{k=1}^{\eta} \Big[M_k\Big(||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \le N < \infty. \tag{8}$$

Suppose that $x = (x_k) \notin \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$. Then for some number $\epsilon_0 > 0$ there is a number s_0

$$||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}|| \geq \epsilon_0, \quad \text{for } s \geq s_0.$$

Therefore

$$[M_k(\epsilon_0)]^{p_k} \leq \Big[M_k\Big(||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k},$$

and hence for each k and s we get

$$\frac{1}{\eta}\sum_{k=1}^{\eta}[M_k(\epsilon_0)]^{p_k} \leq N < \infty,$$

for some $N > 0$, which contradicts (6). Hence

$$\Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m) \subseteq \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m).$$

□

Theorem 3.6. *Suppose* $\mathcal{M} = (M_k)$ *be a Musielak-Orlicz function and let* $1 \leq p_k \leq \sup_k p_k < \infty$. *Then*

$$\Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m) \subseteq \Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$$

holds if and only if

$$\lim_\eta \frac{1}{\eta}\sum_{k=1}^{\eta}[M_k(t)]^{p_k} = 0, \quad t > 0. \tag{9}$$

Proof. Let $\Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m) \subseteq \Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$. Suppose that (9) does not hold. Then for some $t_0 > 0$,

$$\lim_\eta \frac{1}{\eta}\sum_{k=1}^{\eta}[M_k(t)]^{p_k} = L \neq 0. \tag{10}$$

Define $x = (x_k)$ by

$$(x_k) = t\sum_{v=0}^{k-m}(-1)^m \begin{pmatrix} m+k-v-1 \\ k-v \end{pmatrix}$$

for $k = 1, 2, ...$ Then $x = (x_k) \notin \Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$ but $x = (x_k) \in \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m)$. Hence (9) must hold.

Conversely, let $x = (x_k) \in \Big[\hat{c}, u, p, ||\cdot, \cdots, \cdot||\Big]_\infty(\Delta_r^m)$. Then for every k and s, we have

$$||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}|| \leq N < \infty.$$

Therefore

$$\Big[M_k\Big(||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq [M_k(N)]^{p_k}$$

and

$$\lim_\eta \frac{1}{\eta}\sum_{k=1}^{\eta}\Big[M_k\Big(||\frac{u_k \Delta_r^m x_{k+s}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq \lim_\eta \frac{1}{\eta}\sum_{k=1}^{\eta}[M_k(N)]^{p_k} = 0.$$

Hence $x = (x_k) \in \Big[\hat{c}, \mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_0(\Delta_r^m)$. This completes the proof. □

4. Acknowledgement

The authors thank the referee(s) for their valuable suggestions that improved the presentation of the paper.

References

[1] Et, M. and Çolak, R., (1995), On some generalized difference sequence spaces, Soochow J. Math., 21(4), pp.377-386.
[2] Gähler, S., (1963), 2-metrische Raume und ihre topologishe Struktur, Math. Nachr., 26, pp.115-148.
[3] Gunawan, H., (2001), On n-inner product, n-norms, and the Cauchy-Schwartz inequality, Scientiae Mathematicae Japonicae, 5, pp.47-54.
[4] Gunawan, H., (2001), The space of p-summable sequence and its natural n-norm, Bull. Aust. Math. Soc., 64, pp.137-147.
[5] Gunawan, H. and Mashadi, M., (2001), On n-normed spaces, Int. J. Math. Math. Sci., 27, pp.631-639.
[6] Kızmaz, H., (1981), On certain sequence spaces, Cand. Math. Bull., 24(2), pp.169-176.
[7] Lindenstrauss, J. and Tzafriri, L., (1971), On Orlicz seequence spaces, Israel J. Math., 10, pp.379-390.
[8] Lorentz, G.G., (1948), A contribution to the theory of divergent series, Act. Math., 80, pp.167-190.
[9] Maddox, I.J., (1967), Spaces of strongly summable sequences, Quart. J. Math., 18, pp.345-355.
[10] Maddox, I.J., (1978), A new type of convergence, Math. Proc. Camb. Phil. Soc., 83, pp.61-64.
[11] Maligranda, L., (1989), Orlicz Spaces and Interpolation, Seminars in Mathematics 5, Polish Academy of Science.
[12] Misiak, A., (1989), n-inner product spaces, Math. Nachr., 140, pp.299-319.
[13] Musielak, J., (1983), Orlicz Spaces and Modular Spaces, Lecture Notes in Mathematics, 1034p.
[14] Nanda, S., (1984), Strongly almost convergent sequences, Bull. Call. Math. Soc., 76, pp.236-240.
[15] Raj, K., Sharma, A.K. and Sharma, S.K., (2011), A Sequence space defined by Musielak-Orlicz functions, Int. J. Pure Appl. Math., 67, pp.475-484.
[16] Raj, K. and Sharma, S.K., (2011), Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz functions, Acta Univ. Sapientiae Math., 3, pp.97-109.
[17] Raymond, W., Freese, Y. and Cho, J., (2001), Geometry of Linear 2-normed Spaces, N. Y. Nova Science Publishers, Huntington.
[18] Sahiner, A., Gurdal, M., Saltan, S. and Gunawan, H., (2007), Ideal convergence in 2-normed spaces, Taiwanese J. Math., 11(5), pp.1477-1484.
[19] Wilansky, A., (1984), Summability through Functional Analysis, North- Holland Math. Studies, North-Holland Publishing, Amsterdam, 85.

Kuldip Raj is an assistant Professor in the School of Mathematics, Shri Mata Vaishno Devi University, Katra, India. His area of specialization is sequence series, summability and operator theory.

Sunil K. Sharma is a Ph.D. Scholar in the School of Mathematics, Shri Mata Vaishno Devi University, Katra, India. His area of specialization is sequence series, summability and operator theory.